In the case of the system when the base of the cylinder is raised above the plane of the elementary area by an amount y_0 (Fig. 1b) the local angular coefficient is determined from the equation

$$\varphi_{dF_1 \cdot F_2} = \varphi^{*}_{dF_1 \cdot F_2} - \varphi^{**}_{dF_1 \cdot F_2}, \qquad (7)$$

where $\varphi^{*}_{dF_1 \cdot F_2}$ and $\varphi^{**}_{dF_1 \cdot F_2}$ are the local angular coefficients for cylinders of heights y_1 and y_0.

Curves of the dependence of the local angular coefficient on the dimensionless distance $X = x_1/R$ and the dimensionless height $Y = y_1/R$, obtained on the basis of the solutions (5) and (6), are presented in Fig. 2.

NOTATION

dF_1, elementary area; F_2, section of emitting surface seen from center of area dF_1; $\varphi dF_1 dF_2$, local angular coefficient; R, radius of cylinder; x_1, distance from center of area dF_1 to axis of cylinder ($X = x_1/R$, dimensionless distance); y_1 and y_0, height of cylinder ($Y = y/R$, dimensionless height); r, distance of center of area dF_1 from an arbitrary point on surface F_2; N, normal to center of area dF_1; Φ_0, angle defining boundary of visible section of cylindrical surface; θ_1, angle between normal to center of area dF_1 and straight line connecting center of dF_1 with an arbitrary point on surface F_2; θ_2, angle between normal to surface F_2 at an arbitrary point and straight line connecting this point with center of dF_1; x, y, z, coordinates.

LITERATURE CITED

1. E. M. Sparrow and R. D. Cess, Radiant Heat Transfer, Brooks/Cole Publ. Co., Belmont, California (1970).
2. A. G. Blokh, Principles of Radiant Heat Exchange [in Russian], Gos. Énerg. Izd., Moscow–Leningrad (1962).

METHOD OF "JOINING" OF SOLUTIONS IN THE DETERMINATION OF A PLANE AND A CYLINDRICAL PHASE INTERFACE IN THE STEFAN PROBLEM

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It is shown that the position of the phase interface in the Stefan problem can be expressed through two functions: One function determines the position of the melting-temperature isotherm in the problem without phase transitions and the second does not depend on time.

Two most popular courses presently exist for determining the law of motion of the phase interface in the Stefan problem: approximate analytical solutions of the problem and numerical methods using a computer. Considerable difficulties are encountered on the latter course in the stage of analysis of the numerical material obtained and the attempt to represent the results in the form of analytical dependences of the position of the phase interface on the determining parameters and criteria. The following method can be proposed as one variant of analysis. Let the amount of latent heat of the phase transitions per unit volume of material approach zero. In this case a plane interface, for example, approaches its limiting value

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$\xi_0(\tau)$. It is obvious that $\xi_0(\tau)$ characterizes the position of the isotherm T_{me} at the time τ in the ordinary nonsteady problem without phase transitions.

The idea of the method consists in determining the coordinate of the phase interface as a combination of two functions, one of which ($\xi_0(\tau)$) varies with time while the second is a constant quantity for the given problem, determined by the assigned parameters. Thus, for the plane one-dimensional problem we set

$$\xi - a = (\xi_0 - a)\,\psi, \quad a = \text{const}. \tag{1}$$

Analogously, for the cylindrical coordinate we write

$$h - r_0 = (h_0 - r_0)\,\psi_1, \tag{2}$$

since the thawing radius (of the melted zone) is usually measured from the axis of the cylinder. From physical considerations it follows that $0 < \psi < 1$ and $0 < \psi_1 < 1$. Henceforth we will call ψ and ψ_1 the moderating ratios. When Eqs. (1) and (2) are valid the calculation of the phase interfaces comes down to the determination of ψ and ψ_1, since the functions ξ_0 and h_0 are known for many problems [1, 2]. In this connection the method which we proposed for determining the phase interface is called the method of "joining" of solutions. For the most general statement of the Stefan problem it is an approximate method. In a number of cases, however, the method admits of an exact interpretation of the results of a computer calculation, such as with a constant boundary condition of the first kind. The possibility of applying this method for a semibounded uniform space is well illustrated by the well-known exact but particular solutions of Neumann–Stefan [1] and A. V. Lykov [3]. For the process of thawing of frozen rocks $T_{me} = 0°C$. These solutions are written in the following form:

$$\xi = \beta_1 \sqrt{\tau}, \tag{3}$$

$$\frac{\sqrt{\pi}}{2} LW\beta_1 = T_w \frac{\lambda_t}{\sqrt{a_t}} \frac{\exp\left(-\frac{\beta_1^2}{4a_t}\right)}{\operatorname{erf}\left(\frac{\beta_1}{\sqrt{4a_t}}\right)} + T_r \frac{\lambda_M}{\sqrt{a_M}} \frac{\exp\left(-\frac{\beta_1^2}{4a_f}\right)}{\operatorname{erf}\left(\frac{\beta_1}{\sqrt{4a_f}}\right)}, \tag{4}$$

$$\xi = \beta_2 \sqrt{\tau}, \tag{5}$$

$$\frac{\sqrt{\pi}}{2} LW\beta_2 = T_r \left[\frac{\lambda_t}{\sqrt{a_t}} \operatorname{cth}\left(\beta_2 \sqrt{\frac{\pi}{4a_t}}\right) + \frac{\lambda_f}{\sqrt{a_f}}\right] + (T_w - T_r)\frac{\lambda_t}{\sqrt{a_t}} \frac{1}{\operatorname{sh}\left(\beta_2\sqrt{\frac{\pi}{4a_t}}\right)}. \tag{6}$$

Here the entire effect of the phase transitions is taken into account by the left-hand terms of Eqs. (4) and (6). When W = 0 the rate of movement of the 0°C isotherm in the ordinary nonsteady problem without phase transitions will be characterized by β_1 and β_2. The form of the solution is not changed in this case. Unfortunately, the moderating ratio ψ cannot be obtained in explicit form. The quantity ψ can be obtained using the majority of the approximate solutions, which usually have the form

$$(\xi - a)^2 = \frac{f(\tau)}{\psi(W)}. \tag{7}$$

If $\psi(0) = B \neq 0$ (and we are considering just such equations), then in the problem without phase transitions the position of the zero isotherm is determined from the equation

$$(\xi_0 - a)^2 = \frac{f(\tau)}{B}. \tag{8}$$

Then from Eqs. (7) and (8) we obtain

$$\xi - a = (\xi_0 - a)\sqrt{\frac{B}{\psi(W)}} = (\xi_0 - a)\,\psi. \tag{9}$$

We can show that for the similar cylindrical problem one can approximately take

$$\frac{h_0 - r_0}{h - r_0} = \psi_1 = \text{const}. \tag{10}$$

TABLE 1. Values of the Coefficients in Eqs. (19).

Fo	$-k_1$	$-k_2$	$-\bar{k}_1$	$-\bar{k}_2$
0,1—0,3	0,0311	0,362	0,866	1,136
0,3—1,0	0,0355	0,370	0,866	1,136
1—2	0,0355	0,337	0,866	1,303
2—3	0,0472	0,298	0,765	1,639
3—6	0,0223	0,350	0,728	1,714
6—10	0,0355	0,333	—0,226	2,942

For this, with the aim of determining the thawing radius h, we solved the following system of differential equations on an M-222 computer:

$$\frac{\partial T_t}{\partial \tau} = a_t \left(\frac{\partial^2 T_t}{\partial r^2} + \frac{1}{r} \frac{\partial T_t}{\partial r} \right); \quad r_0 \leqslant r < h, \tag{11}$$

$$\frac{\partial T_f}{\partial \tau} = a_f \left(\frac{\partial^2 T_f}{\partial r^2} + \frac{1}{r} \frac{\partial T_f}{\partial r} \right); \quad h \leqslant r \leqslant r_a. \tag{12}$$

$$\left(\lambda_f \frac{\partial T_f}{\partial r} - \lambda_t \frac{\partial T_t}{\partial r} \right)_{r=h} = LW \frac{dh}{d\tau} \tag{13}$$

with the following conditions:

$$T_t (r_0, \tau) = T_w; \quad T_f (h) = 0; \quad T_f (r_r, \tau) = T_r; \quad h(0) = r_0; \quad T_f (r, 0) = T_r .$$

The quantity r_a was taken as equal to twice the effective radius of action r_e [4]: $r_a = 2r_e$; $r_e = r_0 + 2.18\sqrt{a_m \tau}$.

The quantity r_e was calculated at the maximum values of $a_f \tau$ and then was taken as constant. A computation algorithm was developed for the solution of the system (11)-(13). For this we used the difference method of "trapping" the phase front at a node of the difference grid [5]. Its essence consists in the following: The region being studied is divided into a grid with a step Δr = const while the differential equations in partial derivatives are replaced by implicit difference equations. The thawing time per step of the grid is found by the iteration method using an equation derived from the condition (13). Then at each iteration step the systems of difference equations are solved separately for the two zones. A variable time step is obtained in this case. In analyzing the results of the calculations on the M-222 computer we used the theory of the similarity of thermal processes with phase transitions. Without dwelling on the derivation of the similarity criteria, we write the quantities determining the process which we studied:

$$\mathrm{Fo} = \frac{a_t \tau}{r_0^2}; \quad \mathrm{Ko} = \frac{LW}{C_{Vf} T_w}; \quad \mathrm{Kv} = \frac{\lambda_t}{\lambda_f}; \quad -\frac{T_r}{T_w} = \theta; \quad \frac{C_{Vt}}{C_{Vf}} = C_{tf} . \tag{14}$$

We also introduce the dimensionless parameters $H = h/r_0$ and $R = r/r_0$. The values of the criterial quantities and dimensionless parameters were varied in the following ranges:

$$1 < H \leqslant 3.01, \quad 0 < \mathrm{Fo} < 6.7, \quad 0.05 \leqslant \mathrm{Ko} \leqslant 8, \quad 1 \leqslant C_{tf} \leqslant 2, \quad 0.25 \leqslant \mathrm{Kv} \leqslant 1, \quad 0.005 \leqslant \theta \leqslant 0.3. \tag{15}$$

When W = 0 we have $a_t = a_f = a$ and $C_{Vt} = C_{Vf} = C_V$, since in the Stefan problem a change in the thermophysical parameters is accomplished only through phase transitions. To determine the 0°C isotherm, i.e., the limiting value of the thawing radius, it is necessary to solve the equation

$$\frac{\partial V}{\partial \mathrm{Fo}} = \frac{\partial^2 V}{\partial R} + \frac{1}{R} \frac{\partial V}{\partial R}; \quad 1 \leqslant R < \infty, \quad V = \frac{T(R, \mathrm{Fo}) - T_r}{T_w - T_r} \tag{16}$$

with the conditions $V(R, 0) = 0$ and $V_w = v(1, \mathrm{Fo}) = 1$.

The solution of this equation with the given conditions is known [6] but it is expressed through a complicated integral. For small values of Fo (up to 0.2-0.3) this integral is approximated by the following equation [6]:

TABLE 2. Values of ψ_1 for Three Variants: No. 6 (θ = 0.026, Ko = 1.23), No. 18 (θ = 0.026, Ko = 0.31), No. 56 (θ = 0.078, Ko = 0.5).

H	№ 6		№ 18		№ 56	
	Fo	ψ_1	Fo	ψ_1	Fo	ψ_1
1,30	0,068	0,373	0,026	0,601	0,037	0,648
1,42	0,142	0,376	0,053	0,590	0,079	0,620
1,69	0,418	0,369	0,156	0,590	0,236	0,617
1,96	0,860	0,366	0,319	0,585	0,489	0,614
2,14	1,254	0,364	0,463	0,581	0,715	0,613
2,32	1,733	0,363	0,637	0,579	0,991	0,613
2,50	2,300	0,361	0,842	0,578	1,317	0,612
2,68	2,958	0,360	1,079	0,576	1,696	0,612
2,86	3,711	0,357	1,349	0,574	2,130	0,613
3,01	4,413	0,355	1,600	0,573	2,536	0,613

$$V(R,\ \mathrm{Fo}) = \sqrt{\frac{1}{R}}\ \mathrm{erfc}\left(\frac{R-1}{2\sqrt{\mathrm{Fo}}}\right). \tag{17}$$

Values of V = V(R, Fo) are presented in [6] for $0.001 \le \mathrm{Fo} \le 1000$ with different steps in Fo and R. Through an analysis of these data we propose an interpolation equation for $0.1 < \mathrm{Fo} \le 10$:

$$V(R,\ \mathrm{Fo}) = \exp\left[-A(R-1) - B(R-1)^2\right]. \tag{18}$$

The coefficients A and B were determined on the basis of [6] by the method of least squares:

$$A = 10^{k_1}\mathrm{Fo}^{k_2};\quad B = 10^{\bar{k}_1}\mathrm{Fo}^{\bar{k}_2}. \tag{19}$$

The values of k_1, k_2, $\bar{k}_1$, and $\bar{k}_2$ are presented in Table 1. Since the position of the zero isotherm $R = H_0$ is determined from (18) by the condition $T(H_0,\ \mathrm{Fo}) = 0$, we obtain

$$H_0 - 1 = \frac{\sqrt{A^2 - 4B\ln\dfrac{1}{\theta}} - A}{2B}. \tag{20}$$

For the range of $0 < \mathrm{Fo} \le 0.1$ the value of H_0 was determined from Eq. (17) with $R = H_0$. Thus, the function $\psi_1 = \psi_1(\mathrm{Ko}, \theta, \mathrm{Kv}, C_{tf})$ was determined from the relation $(H_0 - 1)/(H - 1) = \psi_1$.

The results of the calculations of the function ψ_1 showed that ψ_1 = const practically for the given variant. The function ψ_1 was determined for a total of 70 variants of the problem. The values of ψ_1 were determined with different values of Fo for each variant. Then the average value of the function ψ_1 was calculated for each variant. The results of the calculation of ψ_1 for three variants are presented in Table 2.

It is difficult to estimate the error in the determination of ψ_1 with different values of H (or Fo) because of the approximate nature of Eqs. (17) and (18).

Thus, Eqs. (1) and (2) can be used in the solution of thermal problems with phase transitions. Finding the moderating ratios using a computer poses no difficulties. It is also of interest to clarify the applicability of Eqs. (1) and (2) with boundary conditions of the second or third kind.

NOTATION

T, r, τ, temperature, cylindrical coordinate, time; C_V, λ, a, volumetric heat capacity, thermal conductivity, and thermal diffusivity of rocks; T_{me}, T_r, T_w, melting temperature, natural temperature of rocks, and temperature at cylinder wall or at surface of semibounded massif of rocks; ξ, h, ξ_0, h_0, thawing depth and radius and their limiting values; r_0, radius of cylinder; L, latent heat of ice melting; W, ice content per unit volume of rocks; Fo, Ko, Kv, Fourier, Kossovich, and Kovner numbers. Indices t and f pertain to thawed and frozen rocks, respectively.

LITERATURE CITED

1. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 2nd ed., Oxford University Press (1959).
2. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
3. A. V. Lykov, Heat Conduction and Diffusion in the Production of Leather, Synthetics, and Other Materials [in Russian], Gizlegprom, Moscow–Leningrad (1941).
4. I. M. Kutasov, Freiberger Forschungshefte, C238, 55-61 (1968).
5. B. M. Budak, F. P. Vasil'ev, and A. V. Uspenskii, Numerical Methods in Phase Dynamics [in Russian], Vol. 4, Vychisl. Tsentr. Mosk. Gos. Univ. (1965), pp. 139-183.
6. J. C. Jaeger, J. Math. Phys., 34, No. 4, 316-321 (1956).

STEADY HEAT TRANSFER TO A THIN INFINITE DISK WITH A CUT-OUT OPENING

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A method is proposed which allows one to find the steady temperature gradient at the rim of a round opening cut out in an infinite nonuniform disk from the given temperature of the rim without a preliminary determination of the temperature field.

A method was proposed earlier which allows one to find the change in the temperature gradient at the boundary of a semiinfinite region from the given change in the temperature of the boundary without a preliminary determination of the temperature field [1, 2]. In the present report the analogous problem is solved for the steady case.

First let us consider the method in application to a well-studied problem.

The steady cylindrically symmetrical temperature field in a uniform infinite disk with a round cut-out opening, cooled from the lateral surface in accordance with Newton's law, is described by the problem

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\cdot\frac{d}{dr}-\gamma\right)T=0;\quad R\leqslant r<\infty; \tag{1}$$

$$T|_{r=R}=T_R;\ T|_{r=\infty}=0;\ \gamma=\text{const}>0.$$

It is required to find the quantity $q_R = (\partial T/\partial r)_{r=R}$, which determines the heat flux to the disk.

The known solution has the form

$$T=T_R\,\frac{K_0(\sqrt{\gamma}\,r)}{K_0(\sqrt{\gamma}\,R)};\ -q_R=\sqrt{\gamma}\,\frac{K_1(\sqrt{\gamma}\,R)}{K_0(\sqrt{\gamma}\,R)}\,. \tag{2}$$

The proposed method of finding q_R without a preliminary determination of the temperature field consists in the following. We represent Eq. (1) in the form of a product of two operators, each of which contain only the first derivative with respect to r:

$$\left[\frac{d}{dr}-\sum_{n=0}^{\infty}\gamma^{\frac{1-n}{2}}\,b_n(r)\right]\left[\frac{d}{dr}+\sum_{n=0}^{\infty}\gamma^{\frac{1-n}{2}}\,a_n(r)\right]T=0. \tag{3}$$

By analogy with [1] the functions a_n and b_n can be determined using recurrent equations if one "multiplies" the operator expressions in brackets and equates terms with equal powers of $\gamma^{-1/2}$ to the original operator (1). It turns out that

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